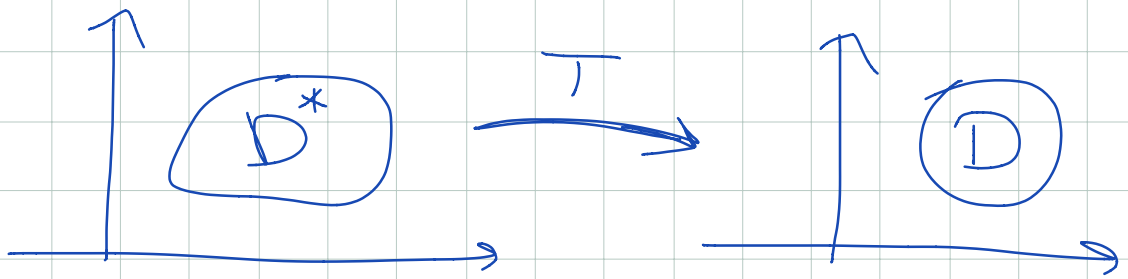
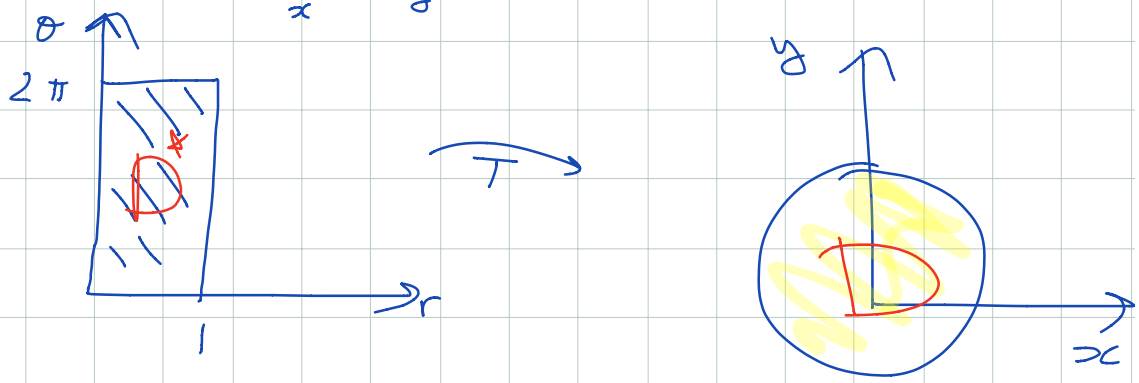
in single variable calculus, many times we needed to do a change of variable to make an integral easier.
We often need to do something similar when we have several variables. A key ingredient is to come up with an appropriate mapping from subsets of $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$.



Example: (Polar coordinates)

Let D^* be defined by $0 \le r \le 1$ & $0 \le \theta \le 2\pi$

$T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$



because $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 1$

so $D = T(D^*) \subset$ Ball of radius one.

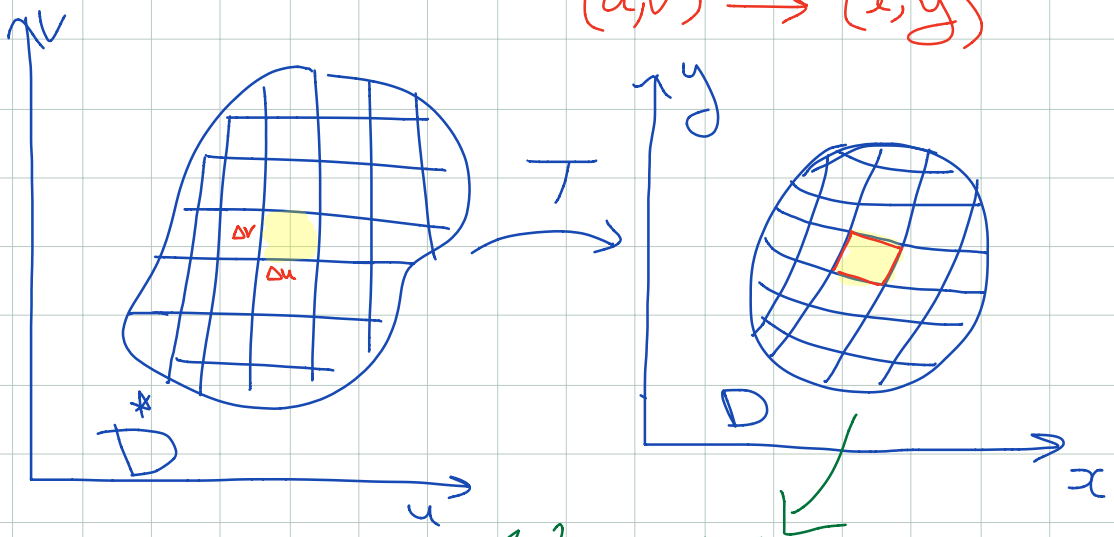
But is it the whole Ball? Yes! Because for any pt (x,y) such that $x^2+y^2 \le 1$ $\exists (r,\theta)$ so that $x = r\cos\theta$, $y = r\cos\theta$.

Question: $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} \boxed{?}\,du\,dv$?

what about in $\mathbb{R}^n$? Let's first stay in $\mathbb{R}^2$ and things will be similar in $\mathbb{R}^n$

use $T: \mathbb{R}^2 \to \mathbb{R}^2$

$(u,v) \to (x,y)$



We want to compute, say the area $A(D)$

The Idea

$A(D)$ = sum of areas of the little "almost parallelograms"

$\approx$ (approximately) sum of areas of parallelograms obtained by linear approx. of T

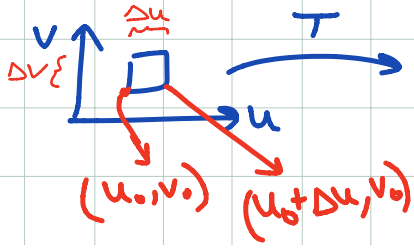
$$A(D) = \iint_D dx\,dy$$

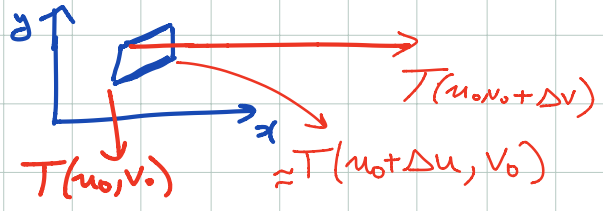
(limit as parallelograms become really small)

So we need the areas of the parallelograms

Linear approximation:

$$T(u,v) \approx T(u_0,v_0) + T'|_{(u_0,v_0)} \begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix}$$





• $T(u_0+\Delta u, v_0) \approx T(u_0,v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}_{(u_0,v_0)} \begin{bmatrix} \Delta u \\ 0 \end{bmatrix} = T(u_0,v_0) + \left(\begin{matrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{matrix} \right)_{(u_0,v_0)} \Delta u$

Similarly, • $T(u_0, v_0+\Delta v) \approx T(u_0+v_0) + \left(\begin{matrix} \partial x/\partial v \\ \partial y/\partial v \end{matrix} \right)_{(u_0,v_0)} \Delta v$

• $T(u_0+\Delta v, v_0+\Delta v) = T(u_0,v_0) + \left(\begin{matrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{matrix} \right) \Delta u + \left(\begin{matrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{matrix} \right) \Delta v$

So the sides of the parallelogram are the vectors

$\frac{\partial x}{\partial u}\Delta u\,\vec{i} + \frac{\partial y}{\partial u}\Delta u\,\vec{j}$

and

$\frac{\partial x}{\partial v}\Delta v\,\vec{i} + \frac{\partial y}{\partial v}\Delta v\,\vec{j}$

→ sides of Parallelogram

Area of the form $= \begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix}$ ← determinant

$= \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right] \Delta u \Delta v$

$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v$

(In general, i.e. in $\mathbb{R}^n$, the volume of an n-parallelogram with sides $\vec{a}_1, \dots, \vec{a}_n$ is $|\det[\vec{a}_1, \dots, \vec{a}_n]|$)

Now, we're ready to find the formula for $A(D)$

$$A(D) \approx \sum_{\text{all } (u_0, v_0)} \sum \text{areas of little parallelograms}$$

$$= \sum_{\text{all } (u_0, v_0)} \sum \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|_{(u_0, v_0)} \Delta u \Delta v$$

$$\xrightarrow{\text{as rect. shrink}} \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\, dv$$

$$\Rightarrow A(D) = \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\, dv$$

In 2D the change of variable formula is

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v),y(u,v)) \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\,dv$$

$D = T(D^*)$

$\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ = Jacobian Determinant

Example: Use Polar Coordinates to find

$$\iint_{\underbrace{x^2+y^2\le 1}_{D}} e^{x^2+y^2}\,dx\,dy$$

Sol'n: ① Find the Jacobian det.

$x = r\cos\theta \qquad y = r\sin\theta$

$$\Rightarrow \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| := \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

② Find D^*

$x^2+y^2 \le 1 \longleftrightarrow 0 \le r \le 1$ & $0 < \theta < 2\pi$ $\rightarrow D^*$

(3) substitute & evaluate

$$\iint_{x^2+y^2\le 1} e^{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2} e^{r^2}\Big|_{r=0}^{1} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} e-1\,d\theta = \pi(e-1)$$

In general, polar coordinates change of variable formula:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Change of Variables formula for triple integrals

Let T be a function $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

with $x = x(u,v,w)$, $y = y(u,v,w)$ & $z = z(u,v,w)$.

Then the Jacobian (determinant) of T, $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

Volume of parallelpiped

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

→ Change of variable formula (3 variable case)

Two Important cases: cylindrical & spherical coordinates

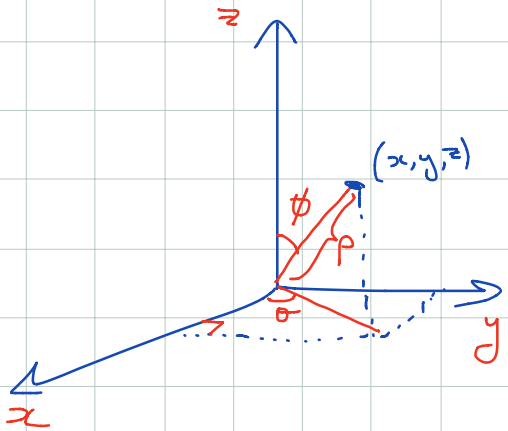
(I) Cylindrical coordinates : $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

So $$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

(II) Spherical coordinates

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$



$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

This is the det. of a 3×3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei-fh) - b(di-gf) + c(dh-ge)$$

In our case $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \cdots = \rho^2\sin\phi$

So: $$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\; d\rho\,d\theta\,d\phi$$

Example Find the volume of the ball of radius R and center $(0,0,0)$ in $\mathbb{R}^3$.

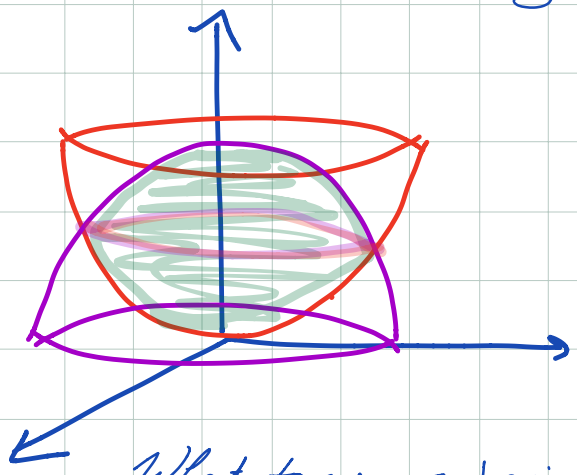
$$V(\text{Ball}) = \iiint_{\text{Ball}} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3}\sin\phi\,d\theta\,d\phi$$

$$= \int_0^{\pi} \frac{2\pi R^3}{3}\sin\phi\,d\phi$$

$$= -\frac{2\pi}{3}R^3(\cos\pi - \cos 0) = \frac{4}{3}\pi R$$

Example: Compute the volume of the solid W between the Paraboloids $z = x^2+y^2$ & $z = 1-x^2-y^2$



What to use, spherical or cylindrical?

$$\iiint_W dV = \iiint_W r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_{r^2}^{1-r^2} r\,dz\,dr\,d\theta$$

$r^2 = z$ & $1-r^2 = z$ intersect at $r^2 = 1-r^2$ i.e. $r = 1/\sqrt{2}$

$$= \int_0^{2\pi}\int_0^{1/\sqrt{2}} \underbrace{(1-r^2-r^2)}_{r-2r^3}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{2}\Big|_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi} \frac{1}{4} - \frac{1}{8}\,d\theta = \frac{\pi}{4}$$

General Change of Variable Theorem

Let $X \subset \mathbb{R}^n$ be compact, with boundary ∂X having volume 0. Let $U \subset \mathbb{R}^n$ be an open set containing X. Let $\underline{\Phi}: U \to \mathbb{R}^n$ be C^1 with Lipschitz derivative and $\underline{\Phi}$ injective on $X - \partial X$ and suppose $[D\underline{\Phi}(x)]$ is invertible on $X - \partial X$

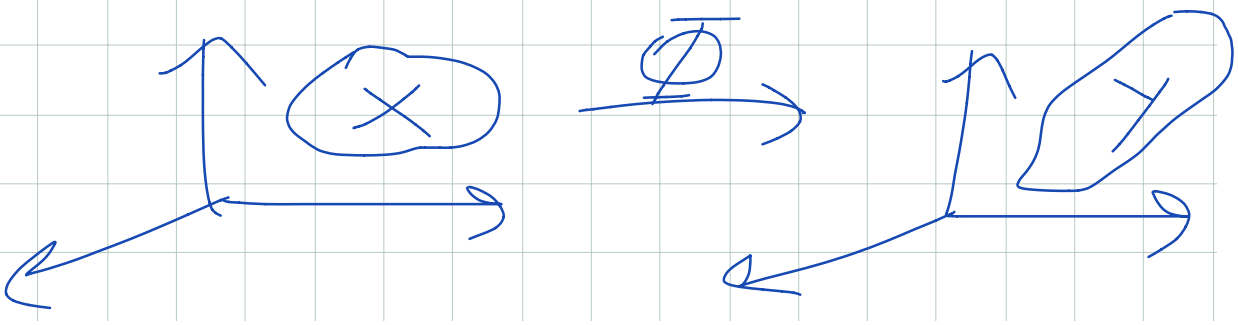
Set $Y = \underline{\Phi}(X)$.

Then if $f: Y \to \mathbb{R}$ is integrable

$(f \circ \underline{\Phi}) \left| \det(D\underline{\Phi}) \right|$ is integrable on X

and

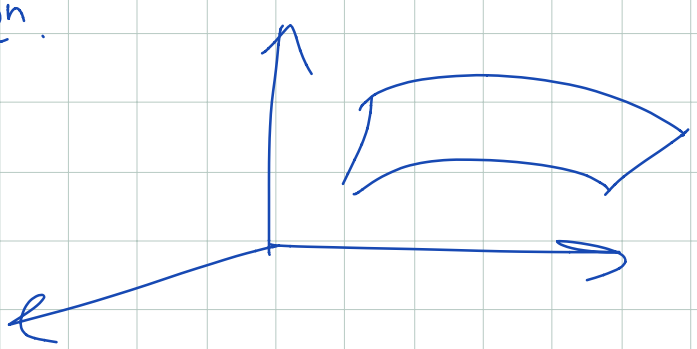
$$\int_Y f(y) \, |d^n y| = \int_X (f \circ \underline{\Phi})(x) \left| \det[D\underline{\Phi}(x)] \right| |d^n x|$$



5) Volumes of manifolds

Suppose we want to calculate the area of a surface, or how much fluid is flowing out of a pipe, or the electric flux through a closed surface.

Let's start by computing the surface area in $\mathbb{R}^3$, and the volume of a k-manifold in $\mathbb{R}^n$.



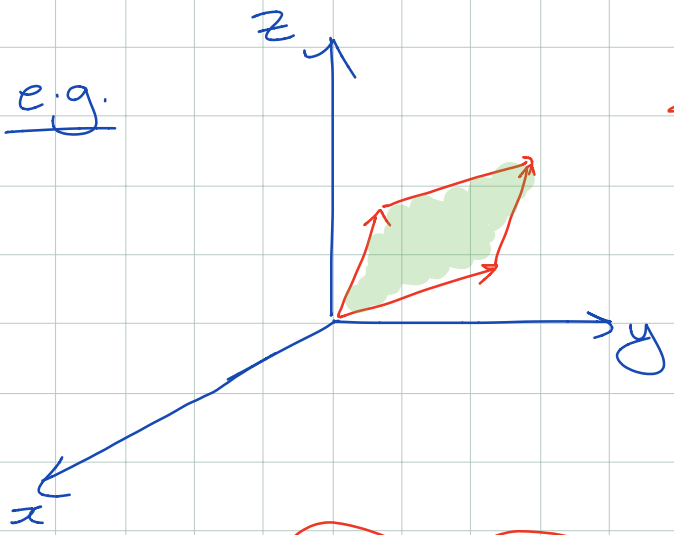
5.1 Volume of Parallelograms

Recall that in $\mathbb{R}^k$, the volume of a k-parm is

$$\text{vol}_k P(\vec{v}_1, \dots, \vec{v}_k) = |\det(\vec{v}_1, \dots, \vec{v}_k)|$$

but what about a k-parm in $\mathbb{R}^n$?

e.g.



2-parm in $\mathbb{R}^3$

Prop: $\mathrm{Vol}_k P(\vec{v}_1, \dots, \vec{v}_k) = \sqrt{\det(T^T T)}$

$T = [\vec{v}_1, \dots, \vec{v}_k]$

Proof: $\det(T^T T) = \det(T^T)\det(T)$

$= [\det(T)]^2$

$\Rightarrow \sqrt{\det(T^T T)} = \det(T)$. ∎

— x —

Now, suppose we have n vectors in $\mathbb{R}^k$ and let $T = [\vec{v}_1, \dots \vec{v}_k] \in \mathbb{R}^{n \times k}$.

Then $T^T T$ is a $k \times k$ matrix, so we can compute its determinant!

Definition: Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, $k \leq n$

Let $T = [\vec{v}_1, \ldots, \vec{v}_k]$

$$\text{Vol}_k P(\vec{v}_1, \ldots, \vec{v}_k) = \sqrt{\det(T^T T)}$$

Example: Let $\vec{v}_1$ & $\vec{v}_2$ be two unit length vectors in $\mathbb{R}^3$, with $\frac{\pi}{3}$ being the angle between them. Find the volume of the parm they span

Soln: $\text{Vol}_k(P) = \sqrt{\det(T^T T)}$

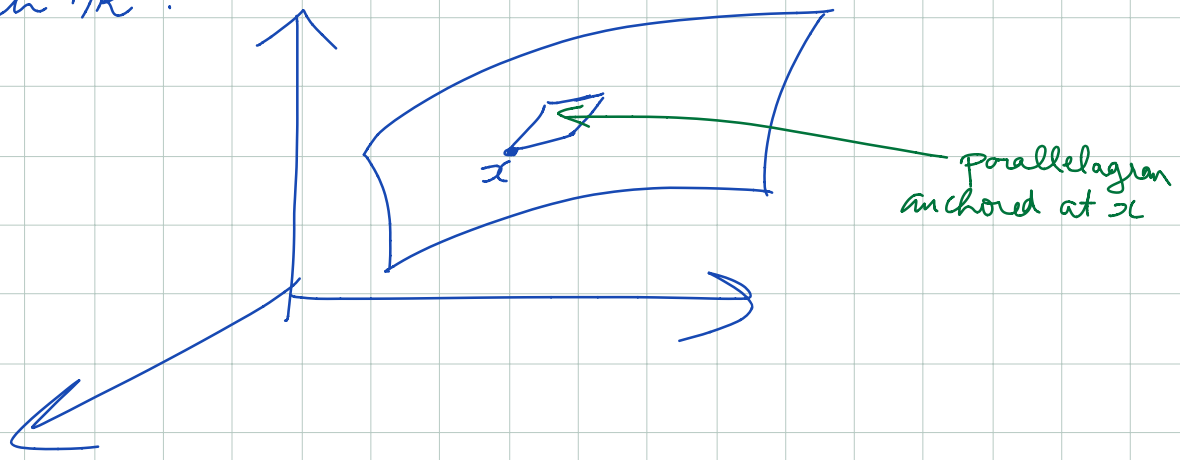
$$T = [\vec{v}_1, \vec{v}_2]$$

$$\Rightarrow T^T T = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix} [\vec{v}_1 \;\; \vec{v}_2] = \begin{bmatrix} \|\vec{v}_1\|^2 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_1 \cdot \vec{v}_2 & \|\vec{v}_2\|^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos \pi/3 \\ \cos \pi/3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$\Rightarrow \sqrt{\det T^T T} = \sqrt{1 - 1/4} = \sqrt{3}/2$$

• Figuring out the area of k-parms in $\mathbb{R}^n$ is the first ingredient in figuring out the volume of a k-manifold in $\mathbb{R}^n$.



The second ingredient is parametrization

5.2 Parametrization

Example:

the map $\gamma: (\theta, \varphi) \longrightarrow \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}$

is a parametrization of the unit sphere. (once we provide a domain for θ & φ.

Prop: If $k > m$ & M is an m-manifold in $\mathbb{R}^n$ and $X \subset M$ is compact, $\mathrm{Vol}_k(X) = 0$

Def'n: Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold and let $U \subset \mathbb{R}^k$ be a subset with boundary of k-dimensional volume $= 0$.

Let $X \subset U$ be such that $U \setminus X$ is open. Then a continuous mapping $\gamma: U \to \mathbb{R}^n$ parametrizes M if

- $\gamma(U) \supset M$
- $\gamma(U \setminus X) \subset M$
- $\gamma: U \setminus X \to M$ is one to one of class C^1 with locally Lipschitz derivative
- $[D\gamma(u)]$ is one to one for all $\vec{u}$ in $U \setminus X$
- X has k-dimensional volume 0, as does $\gamma(X) \cap C$ for any compact $C \subset M$

parametrization

back to our example: M is the unit sphere

the map $\gamma: (\theta, \varphi) \to \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}$

- U is $[0,2\pi]\times[0,\pi]$ and $\gamma(U)=M$
- X is the boundary of U
- $\gamma(U\setminus X)\subset M$

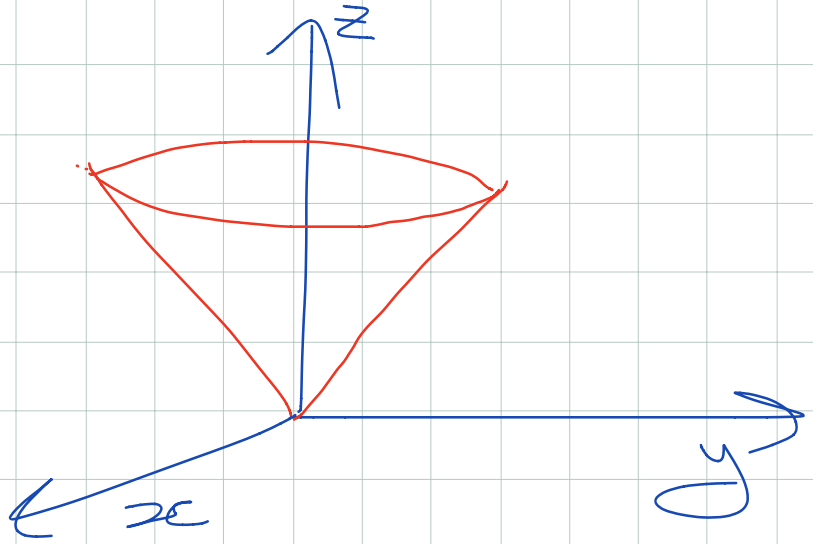
Examples:

$$\{M=(x,y,z)\mid x^2+y^2=z^2,\ z\in(0,1)\}$$

is parametrized by the map

$$\gamma:[0,1]\times[0,2\pi]\longrightarrow\mathbb{R}^3$$
$$(r,\theta)\longmapsto(r\cos\theta, r\sin\theta, r)$$

Exercise: Check that γ satisfies the defn of a parametrization.



Example: Graphs

If U is an open subset of $\mathbb{R}^k$ with boundary ∂U of k-dimensional volume 0 & $f: U \to \mathbb{R}^{n-k}$ is a C^1-mapping, then the graph of f is a manifold in $\mathbb{R}^n$ & the map

$$\vec{x} \longmapsto \begin{pmatrix} \vec{x} \\ f(\vec{x}) \end{pmatrix} \text{ is a parametrization.}$$

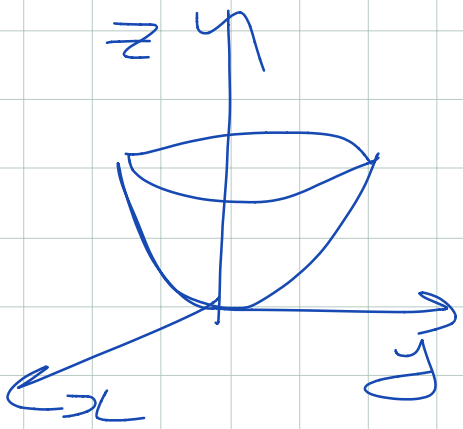
Example

$$\gamma: (x, y) \longrightarrow (x, y, x^2+y^2)$$

parametrizes the surface defined by

$$z = x^2 + y^2$$

(once we specify a domain)



Notice that we could have seen this surface as the graph of $x = \sqrt{z}$ ($f(z)$) rotated about the z-axis

so that points on the surface satisfy $x^2 + y^2 = z$ ($f(z)^2$) (why?)

Then, we can also parametrize the surface by

$$(z, \theta) \mapsto \begin{pmatrix} f(z)\cos\theta \\ f(z)\sin\theta \\ z \end{pmatrix}$$

In our case, this would be $\begin{pmatrix} \sqrt{z}\cos\theta \\ \sqrt{z}\sin\theta \\ z \end{pmatrix}$

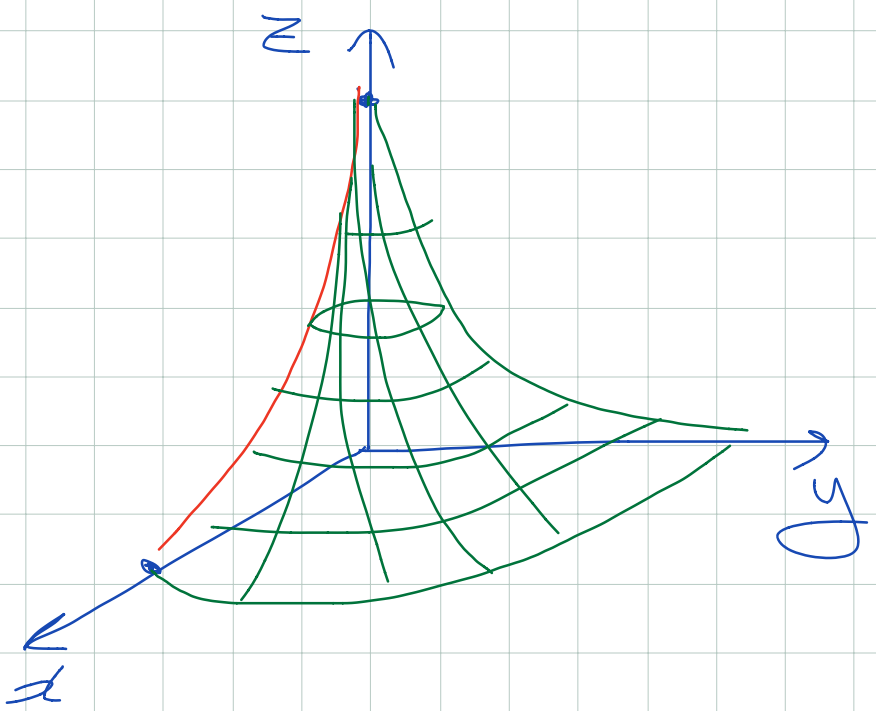
(once we specify the domain)

Example: Surface obtained by rotating a parametrized curve

Let S be the surface obtained by rotating the graph of

$(1-x)^3 = z^2$ in the (x,z) plane where $0 \leq x \leq 1$

about the z-axis



① We can parametrize the curve (not the surface) by $t \to \begin{pmatrix} 1-t^2 \\ t^3 \end{pmatrix}$ where $t \in [0,1]$

② We can parametrize the surface

by $(t,\theta) \longrightarrow \left((1-t^2)\cos\theta, (1-t^2)\sin\theta, t^3\right)$

t^3: untouched because we rotated about z

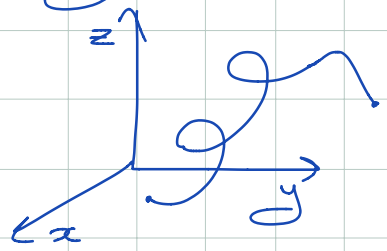
where

$(t,\theta) \in [0,1] \times [0,2\pi]$

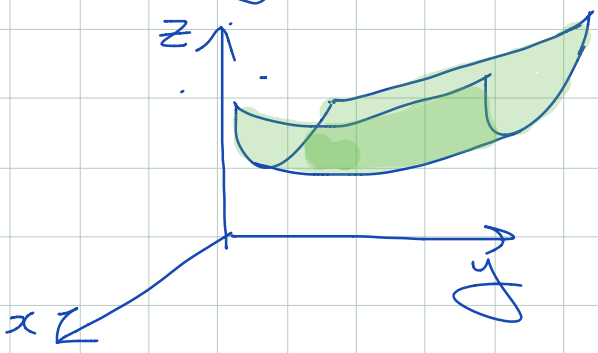
Computing Volumes of Manifolds (5.3)

Goal: We'd like a general formula so we can compute quantities like

- length of a curve in $\mathbb{R}^3$



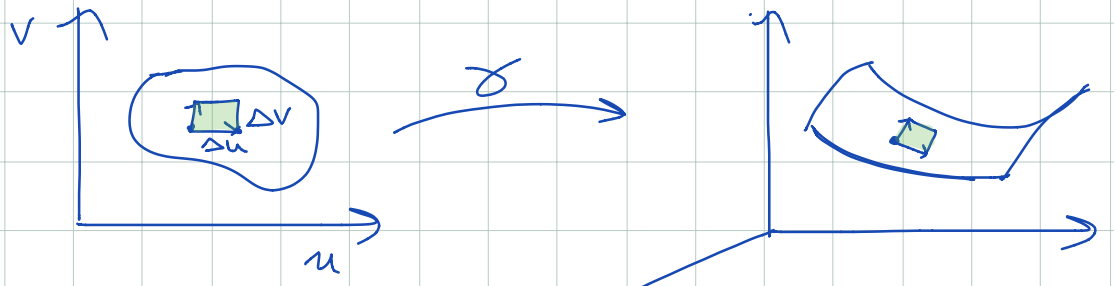
- area of a surface in $\mathbb{R}^3$



• Volume of a k-manifold in $\mathbb{R}^n$.

Basic Idea:

Start with a parametrization of the surface



"Volume of surface ≈ sum of volumes of little parallelograms"

$\longrightarrow$ lims as parms get smaller

Definition: (Volume of k-Manifold in $\mathbb{R}^n$)

Let $M \subset \mathbb{R}^n$ be a smooth k-dimensional manifold & $\gamma: U \to M$ be a parametrization (with X as in the def'n of parametrization)

$$\mathrm{Vol}_k(M) = \int_{U-X} \sqrt{\det\left[D\gamma(u)\right]^T \left[D\gamma(u)\right]}\, |d^k u|$$

$$= \int_{\gamma(U-X)} |d^k x|$$

Example: Let $\gamma : [a,b] \to C$ be the parametrization of a curve

$$\text{arclength} = \int_C |d^1x| = \int_{[a,b]} \sqrt{\det(\gamma'(t) \cdot \gamma'(t))}\,|dt|$$

$$= \int_{[a,b]} |\gamma'(t)|\,dt$$

More generally:

Definition: (Integral with respect to Volume)

Let $M \subset \mathbb{R}^n$ be a smooth k-manifold, and let U & X be as before. Let $\gamma : U \to M$ be a parametrization of M.

Then $f : M \to \mathbb{R}$ is integrable over M w.r.t volume if

$$\int_{U-X} f(\gamma(u)) \sqrt{\det([D\gamma(u)]^T [D\gamma(u)])}\,|d^k u|$$

exists. Moreover

$$\int_M f(x)\,d^k x = \int_{U-X} f(\gamma(u)) \sqrt{\det([D\gamma(u)]^T [D\gamma(u)])}\,|d^k u|$$

Remark: For our definitions to make sense the integral has to be independent of parametrization

Proposition: If $\gamma_1 : U \to M$, $\gamma_2 : V \to M$ are two parametrizations of M

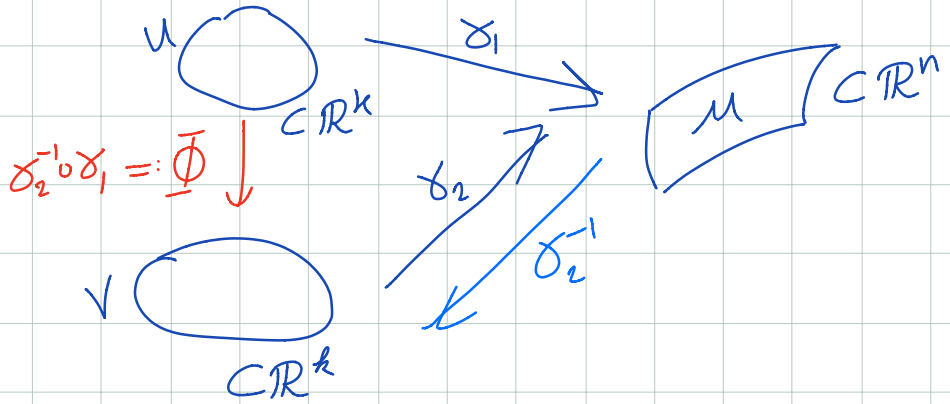
$$\int_U f(\gamma_1(u)) \sqrt{\det\left([D\gamma_1(u)]^T [D\gamma_1(u)]\right)}\, |d^k u|$$

exists if & only if

$$\int_V f(\gamma_2(v)) \sqrt{\det\left([D\gamma(v)]^T [D\gamma(v)]\right)}\, |d^k v|$$

exists, and in that case they are equal

Proof: Let $\Phi : \gamma_2^{-1} \circ \gamma_1$ be the change of parameters map



So $\gamma_1 = \gamma_2 \circ \underbrace{\gamma_2^{-1} \circ \gamma_1}_{\bar{\Phi}}$

and by the chain rule

$$[D\gamma_1(u)] = [D(\gamma_2 \circ \bar{\Phi}(u))]$$

$$= [D\gamma_2|_{\bar{\Phi}(u)}][D\bar{\Phi}|_{(u)}] .$$

Now,

$$\int_V f(\gamma_2(\underbrace{v}_{\bar{\Phi}(u)})) \sqrt{\det([D\gamma_2|_v]^T[D\gamma_2|_v]}\, |d^k v|$$

Change of variable formula

$$= \int_U f(\gamma_1(u)) \sqrt{\det([D\gamma_2|_{\bar{\Phi}(u)}]^T[D\gamma_2|_{\bar{\Phi}(u)}]}\, |\det[D\bar{\Phi}|_u]|\, |d^k u|$$

$$= \int_U f(\gamma_1(u)) \sqrt{\det\left(\left[D\gamma_2 \big|_{\bar\Phi(u)} \right]^T \left[D\gamma_2 \big|_{\bar\Phi(u)} \right] \right)} \sqrt{\det\left(\left[D\bar\Phi \big|_u \right]^T \left[D\bar\Phi \big|_u \right] \right)} \, |d^k u|$$

$$= \int_U f(\gamma_1(u)) \sqrt{\det\left(\underbrace{\left[D\bar\Phi \big|_u \right]^T \left[D\gamma_2 \big|_{\bar\Phi(u)} \right]^T}_{\left[D(\gamma_2 \circ \bar\Phi) \big|_u \right]^T} \right)} \sqrt{\det\left(\underbrace{\left[D\gamma_2 \big|_{\bar\Phi(u)} \right] \left[D\bar\Phi \big|_u \right]}_{\left[D(\gamma_2 \circ \bar\Phi) \big|_u \right]} \right)} \, |d^k u|$$

γ_1

(chain rule)

$$= \int_U f(\gamma_1(u)) \sqrt{\left[D\gamma_1 \big|_u \right]^T \left[D\gamma_1 \big|_u \right]} \, |d^k u|$$



Example:

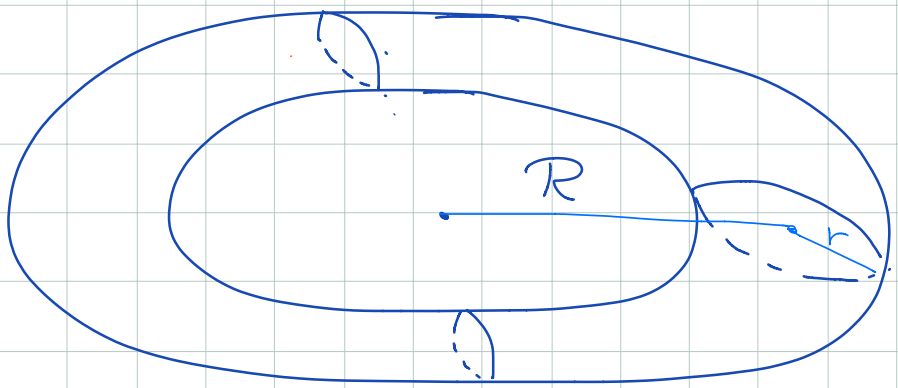
Area of a Torus: Let $R > r > 0$

$$\gamma\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r \sin u \end{pmatrix}$$

where $(u,v) \in [0, 2\pi] \times [0, 2\pi]$

is a parametrization of a torus

(donut)



So, $D\gamma|_{(u,v)} = \begin{bmatrix} -r\sin u\cos v & -(R+r\cos u)\sin v \\ -r\sin u\sin v & -(R+r\cos u)\cos v \\ r\cos u & 0 \end{bmatrix}$

$$\Rightarrow \sqrt{\det\left[D\gamma|_{(u,v)}\right]^T\left[D\gamma|_{(u,v)}\right]} = \sqrt{\det\begin{bmatrix} r^2 & 0 \\ 0 & (R+r\cos u)^2 \end{bmatrix}}$$

$$= r(R+r\cos u)$$

$$\Rightarrow A(\text{Torus}) = \iint_{[0,2\pi]\times[0,2\pi]} r(R+r\cos u)\,|du\,dv|$$

$$= \int_0^{2\pi}\int_0^{2\pi} r(R+\cos u)\,du\,dv = 4\pi^2 rR$$